Entropic uncertainty relations for extremal unravelings of super-operators

Alexey E. Rastegin
Department of Theoretical Physics, Irkutsk State University, Gagarin Bv. 20, Irkutsk 664003, Russia

A way to pose the entropic uncertainty principle for trace-preserving super-operators is presented. It is based on the notion of extremal unraveling of a super-operator. For given input state, different effects of each unraveling result in some probability distribution at the output. As it is shown, all Tsallis' entropies of positive order as well as some of Rényi's entropies of this distribution are minimized by the same unraveling of a super-operator. Entropic relations between a state ensemble and the generated density matrix are revisited in terms of both the adopted measures. Using Riesz's theorem, we obtain two uncertainty relations for any pair of generalized resolutions of the identity in terms of the Rényi and Tsallis entropies. The inequality with Rényi's entropies is an improvement of the previous one, whereas the inequality with Tsallis' entropies is a new relation of a general form. The latter formulation is explicitly shown for a pair of complementary observables in a d-level system and for the angle and the angular momentum. The derived general relations are immediately applied to extremal unravelings of two super-operators.

PACS numbers: 03.65.Ta, 03.67.-a, 02.10.Yn
Keywords: uncertainty principle, Kraus operators, ensemble of quantum states, resolution of the identity

I. INTRODUCTION

Since the famous Heisenberg's paper [1] had been published, uncertainty relations are the subject of long researches [2, 3]. A new interest was stimulated by recent advances in quantum information processing. There are two well-known approaches to formulating the uncertainty principle. The first was initiated by Robertson [4] who showed that a product of the standard deviations of two observables is bounded from below. Here we may run across some disputable topics such as the number-phase case [5]. Both the well-defined Hermitian operator of phase and corresponding number-phase uncertainty relation have been fit within the Pegg-Barnett formalism [6]. The second approach is generally characterized by posing the uncertainty principle via information-theoretic terms especially via entropies [8, 9]. Although the first relation of such a kind was derived by Hirschman [7], a general statement of the problem is examined in the papers [10, 11]. Mutually unbiased bases [14, 15], the time-energy case [16] and tomographic processes [17] have been considered within an entropic approach as well.

The most of known uncertainty relations deals with observables or, more generally, with POVM measurements. Nevertheless, there exist relations for unitary transformations [12] and non-Hermitian annihilation operator [13]. Both the measurement and unitary evolution are rather simplest types of a state change in quantum theory [18]. The formalism of quantum operations, or super-operators, is now a standard tool for treating quantum processes. In the present work, we address a question how to formulate the uncertainty principle for super-operators. It turns out that one of possible ways is naturally provided with the notion of extremal unraveling of a super-operator. For the Shannon entropy, this notion was examined in [19]. As an entropic measure, we will use the Tsallis entropy, which has found use in various physical problems (see references in [20]), and the Rényi entropy.

The paper is organized as follows. In Section II, the properties of the Tsallis entropies and super-operators are recalled. For given input state and super-operator, we find the extremal unraveling that minimizes all the Tsallis entropies simultaneously. Relations between the ensemble entropy and the entropy of a generated density matrix are revisited. In Section III, we derive the uncertainty relation for two generalized resolutions of the identity in terms of Tsallis entropies. The previous result on the Rényi entropies is refined. The obtained entropic relations are directly used for the extremal unravelings of two super-operator. Section IV concludes the paper with a summary of results.

II. TSALLIS' ENTROPIES AND EXTREMAL UNRAVELINGS

First, we briefly recall the definitions of used entropic measures (for a discussion and further references, see [21]). For real $\alpha > 0$ and $\alpha \neq 1$, we define the non-extensive α-entropy of probability distribution $\{p_i\}$ by [22]

$$H_\alpha(p_i) \triangleq (1-\alpha)^{-1} \left(\sum\nolimits_i p_i^\alpha - 1 \right) \ . \tag{2.1}$$

This can be rewritten as $H_\alpha(p_i) = -\sum_i p_i^\alpha \ln_\alpha p_i$, where $\ln_\alpha x \equiv \bigl(x^{1-\alpha} - 1\bigr)/(1-\alpha)$ is the α-logarithmic function, defined for $\alpha \geq 0$, $\alpha \neq 1$ and $x \geq 0$. The quantity (2.1) will be referred to as "Tsallis α-entropy", though it was

previously discussed by Havrda and Charvát [23]. In the limit $\alpha \to 1$, $\ln_\alpha x \to \ln x$ and the quantity (2.1) recovers the Shannon entropy. We will also use the Rényi α-entropy defined for $\alpha \neq 1$ as [24]

$$R_\alpha(p_i) \triangleq (1-\alpha)^{-1} \ln\Bigl(\sum\nolimits_i p_i^\alpha\Bigr) \ . \tag{2.2}$$

The Rényi α-entropy also coincides with the Shannon entropy in the limit $\alpha \to 1$. There are many various forms of extrapolation between different entropies [25]. We will only need in the equality

$$R_\alpha(p_i) = (1-\alpha)^{-1} \ln\Bigl(1 + (1-\alpha) H_\alpha(p_i)\Bigr) \ , \tag{2.3}$$

which immediately follows from the definitions (2.1) and (2.2).

In the following, some notation of linear algebra will be used. Let $\mathcal{H}$ be finite-dimensional Hilbert space. For given two vectors $\boldsymbol{\psi}, \boldsymbol{\varphi} \in \mathcal{H}$, their inner product is denoted by $\langle \boldsymbol{\psi} , \boldsymbol{\varphi} \rangle$. For two operators X and Y on $\mathcal{H}$, we define the Hilbert–Schmidt inner product by [26]

$$\langle \mathsf{X} , \mathsf{Y} \rangle_{\mathrm{hs}} \triangleq \mathrm{tr}(\mathsf{X}^\dagger \mathsf{Y}) \ . \tag{2.4}$$

This inner product naturally induces so-called Frobenius norm $\|\mathsf{X}\|_F = \langle \mathsf{X} , \mathsf{X} \rangle_{\mathrm{hs}}^{1/2}$. The Frobenius norm can be re-expressed as $\|\mathsf{X}\|_F = \Bigl(\sum_j s_j(\mathsf{X})^2\Bigr)^{1/2}$, where the singular values $s_j(\mathsf{X})$ are eigenvalues of $|\mathsf{X}| = \sqrt{\mathsf{X}^\dagger \mathsf{X}}$. The largest singular value of X gives the spectral norm $\|\mathsf{X}\|_\infty$ of this operator [26].

A quantum measurement is described by a "generalized resolution of the identity" (or by a "positive operator-valued measure"). This is a set $\{\mathsf{M}_i\}$ of positive semidefinite operators obeying the completeness relation $\sum_i \mathsf{M}_i = \mathbb{I}$, where $\mathbb{I}$ is the identity [27]. Consider a linear map \$ that takes linear operators on $\mathcal{H}$ to linear operators on $\mathcal{H}'$, and also satisfies the conditions of trace preservation and complete positivity. Following [28], this map will be referred to as "super-operator". Each super-operator has a Kraus representation, namely [18, 28]

$$\$(\rho) = \sum\nolimits_i \mathsf{A}_i \, \rho \, \mathsf{A}_i^\dagger \ , \tag{2.5}$$

where the Kraus operators A_i map the input space $\mathcal{H}$ to the output space $\mathcal{H}'$ and obey $\sum_i \mathsf{A}_i^\dagger \mathsf{A}_i = \mathbb{I}$ (the preservation of the trace). Representations of such a kind are never unique [26]. For given super-operator \$, there are many sets $\mathcal{A} = \{\mathsf{A}_i\}$ that enjoy (2.5). In the paper [19], each concrete set $\mathcal{A} = \{\mathsf{A}_i\}$ resulting in (2.5) is called an "unraveling" of the super-operator \$. This terminology is due to Carmichael [29] who introduced this word for a representation of the master equation. It is well-known that two Kraus representations of the same super-operator are related as

$$\mathsf{B}_i = \sum\nolimits_j \mathsf{A}_j \, u_{ji} \ , \tag{2.6}$$

where the matrix $\mathsf{U} = [[u_{ij}]]$ is unitary [28]. [We assume in (2.6) that unravelings $\mathcal{A}$ and $\mathcal{B}$ have the same cardinality by adding zero operators, if needed.] For given density operator ρ and super-operator unraveling $\mathcal{A} = \{\mathsf{A}_j\}$, we introduce the matrix

$$\mathsf{\Pi}(\mathcal{A}|\rho) \triangleq [[\langle \mathsf{A}_i \sqrt{\rho} , \mathsf{A}_j \sqrt{\rho} \rangle_{\mathrm{hs}}]] \equiv [[\mathrm{tr}(\mathsf{A}_i^\dagger \mathsf{A}_j \rho)]] \ . \tag{2.7}$$

The diagonal element $p_i = \mathrm{tr}(\mathsf{A}_i^\dagger \mathsf{A}_i \rho)$ is clearly positive and gives the ith effect probability. Then the entropies $H_\alpha(\mathcal{A}|\rho)$ and $R_\alpha(\mathcal{A}|\rho)$ are merely defined by (2.1) and (2.2) respectively. By definition, the matrix $\mathsf{\Pi}(\mathcal{A}|\rho)$ is Hermitian. Suppose that the two sets $\mathcal{A} = \{\mathsf{A}_i\}$ and $\mathcal{B} = \{\mathsf{B}_j\}$ fulfill (2.6). Using the properties of the inner product, we have

$$\langle \mathsf{B}_i \sqrt{\rho} , \mathsf{B}_k \sqrt{\rho} \rangle_{\mathrm{hs}} = \sum\nolimits_{jl} u_{ji}^* \, u_{lk} \, \langle \mathsf{A}_j \sqrt{\rho} , \mathsf{A}_l \sqrt{\rho} \rangle_{\mathrm{hs}} \ , \tag{2.8}$$

or $\mathsf{\Pi}(\mathcal{B}|\rho) = \mathsf{U}^\dagger \, \mathsf{\Pi}(\mathcal{A}|\rho) \, \mathsf{U}$ as the matrix relation. That is, if the sets $\mathcal{A}$ and $\mathcal{B}$ are both unravelings of the same super-operator then the matrices $\mathsf{\Pi}(\mathcal{A}|\rho)$ and $\mathsf{\Pi}(\mathcal{B}|\rho)$ are unitarily similar. Due to Hermiticity, all the matrices of a kind $\mathsf{\Pi}(\mathcal{A}|\rho)$ assigned to one and the same super-operator are unitarily similar to a unique (up to permutations) diagonal matrix $\mathsf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$, where the λ_i's and perhaps zeros are the eigenvalues of each of these matrices. So any $\mathsf{\Pi}(\mathcal{A}|\rho)$ is positive semidefinite.

For given unraveling $\mathcal{A} = \{\mathsf{A}_i\}$, we obtain the concrete matrix $\mathsf{\Pi}(\mathcal{A}|\rho)$ and diagonalize it through a unitary transformation $\mathsf{V}^\dagger \, \mathsf{\Pi}(\mathcal{A}|\rho) \, \mathsf{V} = \mathsf{D}$. Let us define a specific unraveling $\mathcal{A}_\rho^{(ex)}$ related to given $\mathcal{A}$ as

$$\mathsf{A}_i^{(ex)} = \sum\nolimits_j \mathsf{A}_j \, v_{ji} \ , \tag{2.9}$$

where the unitary matrix V diagonalizes $\mathsf{\Pi}(\mathcal{A}|\rho)$. It turns out that the unraveling (2.9) enjoys the *extremality* property with respect to all the Tsallis α-entropies for $\alpha \in (0;\infty)$ and the Rényi α-entropies for $\alpha \in (0;1)$.

Theorem II.1. *For given density operator ρ and super-operator \$, each unraveling $\mathcal{A}$ of the \$ satisfies*

$$H_\alpha(\mathcal{A}_\rho^{(ex)}|\rho) \leq H_\alpha(\mathcal{A}|\rho) \quad \forall\ \alpha \in (0;\infty)\ , \tag{2.10}$$

$$R_\alpha(\mathcal{A}_\rho^{(ex)}|\rho) \leq R_\alpha(\mathcal{A}|\rho) \quad \forall\ \alpha \in (0;1)\ , \tag{2.11}$$

where the extremal unraveling $\mathcal{A}_\rho^{(ex)}$ is defined by the formula (2.9).

Proof We firstly note that $\Pi(\mathcal{A}_\rho^{(ex)}|\rho) = \mathsf{D}$ with the probabilities λ_j of effects. In view of $\Pi(\mathcal{A}|\rho) = \mathsf{V}\,\mathsf{D}\,\mathsf{V}^\dagger$, the probabilities $p_i = \mathrm{tr}(\mathsf{A}_i^\dagger \mathsf{A}_i \rho)$ of different effects of the unraveling $\mathcal{A}$ is related to the λ_j's by

$$p_i = \sum\nolimits_j v_{ij}\, \lambda_j\, v_{ij}^* = \sum\nolimits_j s_{ij}\, \lambda_j\ , \tag{2.12}$$

where $s_{ij} = v_{ij}\, v_{ij}^*$. The matrix $\mathsf{S} = [[s_{ij}]]$ is unistochastic, whence $\sum_i s_{ij} = 1$ for all j and $\sum_j s_{ij} = 1$ for all i. We now use these relations and an obvious fact that for $\alpha > 0$ the function $h_\alpha(x) = \big(x^\alpha - x\big)/(1-\alpha)$ is concave in the range $x \in [0;1]$. According to Jensen's inequality (see, e.g., the book [30]), there holds

$$H_\alpha(\mathcal{A}|\rho) = \sum\nolimits_i h_\alpha\Big(\sum\nolimits_j s_{ij}\,\lambda_j\Big) \geq \sum\nolimits_i \sum\nolimits_j s_{ij}\, h_\alpha(\lambda_j) = \sum\nolimits_j h_\alpha(\lambda_j) = H_\alpha(\mathcal{A}_\rho^{(ex)}|\rho)\ . \tag{2.13}$$

This completes the proof for (2.10). Further, we note that for $\alpha < 1$ the function $(1-\alpha)^{-1}\ln\big(1 + (1-\alpha)\,x\big)$ is increasing in the range $x \in [0;\infty)$. Combining the equality (2.3) with (2.10) then gives (2.11). $\square$

For the Shannon entropy, a problem of "minimal" unraveling was considered in [31] and later in [19]. Diagonalizing the matrix $\Pi(\mathcal{A}|\rho)$ is actually equivalent to the extreme condition that has been derived by the method of Lagrange's multipliers in [19]. Latter reasons local in spirit are quite complemented by the above proof based on the concavity. Our treatment allows to find the extremal unraveling easily from (2.9). Thus, for prescribed state ρ all the Tsallis entropies of order $\alpha \in (0;\infty)$ and the Rényi entropies of order $\alpha \in (0;1)$ are minimized by one and the same unraveling of given super-operator. However, this unraveling does not minimize other Rényi entropies in general. An unraveling extremal with respect to Rényi's entropy of order $\alpha > 1$ may also be dependent on α. A search for such an unraveling seems to be difficult since Rényi's entropy of such an order is not purely convex nor purely concave.

The relations (2.10) and (2.11) can be put in the context of a state ensemble having a prescribed density operator. In line with (2.1) and (2.2), we introduce the quantum Tsallis and Rényi entropies of a density matrix ρ by

$$\mathrm{H}_\alpha(\rho) \triangleq (1-\alpha)^{-1}\mathrm{tr}\big(\rho^\alpha - \rho\big)\ , \qquad \mathrm{R}_\alpha(\rho) \triangleq (1-\alpha)^{-1}\ln\big(\mathrm{tr}(\rho^\alpha)\big)\ . \tag{2.14}$$

In the limit $\alpha \to 1$ both the expressions coincides with the von Neumann entropy $-\mathrm{tr}\big(\rho\ln\rho\big)$. Let $\{p_i, \boldsymbol{\psi}_i\}$ be an ensemble of normalized pure states, $\langle\boldsymbol{\psi}_i\,,\boldsymbol{\psi}_i\rangle = 1$ and $\sum_i p_i = 1$, that leads to the density operator ρ, namely

$$\sum\nolimits_i p_i\, \boldsymbol{\psi}_i\, \boldsymbol{\psi}_i^\dagger = \rho = \sum\nolimits_j \lambda_j\, \boldsymbol{\varphi}_j\, \boldsymbol{\varphi}_j^\dagger\ . \tag{2.15}$$

In general, the states $\boldsymbol{\psi}_i$ are not mutually orthogonal. The right-hand side of (2.15) poses the spectral decomposition of ρ, so that the vectors $\boldsymbol{\varphi}_j$ form an orthonormal basis in $\mathcal{H}$. The ensemble classification theorem says that [32]

$$\sqrt{p_i}\,\boldsymbol{\psi}_i = \sum\nolimits_j u_{ij}\sqrt{\lambda_j}\,\boldsymbol{\varphi}_j \tag{2.16}$$

for some unitary matrix $[[u_{ij}]]$. It follows from (2.16) and $\langle\boldsymbol{\varphi}_j\,,\boldsymbol{\varphi}_k\rangle = \delta_{jk}$ that $p_i = \sum_j s_{ij}\,\lambda_j$, where $s_{ij} = u_{ij}^* u_{ij}$ are elements of a unistochastic matrix. Changing notation in the formula (2.13) appropriately, we finally get

$$\mathrm{H}_\alpha(\rho) = H_\alpha(\lambda_j) \leq H_\alpha(p_i) \quad (0<\alpha)\ , \qquad \mathrm{R}_\alpha(\rho) = R_\alpha(\lambda_j) \leq R_\alpha(p_i) \quad (0<\alpha<1)\ . \tag{2.17}$$

In the limit $\alpha \to 1$, both the inequalities are reduced to $-\mathrm{tr}\big(\rho\ln\rho\big) \leq H_1(p_i)$. The last is a known relation between the von Neumann entropy of the generated density operator and the Shannon entropy of an ensemble [33]. For the Tsallis entropies this result can be proceeded to ensembles of mixed states. Let an ensemble $\{p_i, \omega_i\}$ of normalized density operators, $\mathrm{tr}(\omega_i) = 1$ and $\sum_i p_i = 1$, give rise to a density operator

$$\rho = \sum\nolimits_i p_i\,\omega_i = \sum\nolimits_{ij} p_i\,\nu_{ij}\,\boldsymbol{\varphi}_{ij}\,\boldsymbol{\varphi}_{ij}^\dagger\ , \tag{2.18}$$

where the spectral decomposition $\omega_i = \sum_j \nu_{ij}\, \boldsymbol{\varphi}_{ij}\, \boldsymbol{\varphi}_{ij}^{\dagger}$. Applying the first inequality of (2.17) to the right-hand side of (2.18), we have

$$\mathrm{H}_\alpha(\rho) \leq -\sum_{ij} p_i^\alpha \nu_{ij}^\alpha \ln_\alpha(p_i \nu_{ij}) = -\sum_{ij} p_i^\alpha \nu_{ij}^\alpha \bigl(\ln_\alpha \nu_{ij} + \nu_{ij}^{1-\alpha} \ln_\alpha p_i\bigr) = -\sum_i p_i^\alpha \sum_j \nu_{ij}^\alpha \ln_\alpha \nu_{ij} - \sum_i p_i^\alpha \ln_\alpha p_i \ .$$

Here we used the identity $\ln_\alpha(xy) \equiv \ln_\alpha y + y^{1-\alpha} \ln_\alpha x$. Adding a consequence of concavity of the function $h_\alpha(x) = \bigl(x^\alpha - x\bigr)/(1-\alpha)$, we finally write

$$\sum\nolimits_i p_i\, \mathrm{H}_\alpha(\omega_i) \leq \mathrm{H}_\alpha(\rho) \leq \sum\nolimits_i p_i^\alpha\, \mathrm{H}_\alpha(\omega_i) + H_\alpha(p_i) \ . \tag{2.19}$$

The inequality on the left can be shown as follows. For any concave function $f(x)$, the functional $\mathrm{tr}\bigl(f(\mathsf{X})\bigr)$ is also concave on Hermitian X (for details, see section III in [34]). We further note that $\mathrm{H}_\alpha(\rho) = \mathrm{tr}\bigl(h_\alpha(\rho)\bigr)$. In the limit $\alpha \to 1$, the inequalities (2.19) are reduced to the well-known bounds on the von Neumann entropy of a state mixture (for instance, see [33]). It seems that such a treatment fails in the case of Rényi's entropies. Indeed, the Tsallis α-entropy does enjoy so-called strong additivity of degree α, whereas the Rényi α-entropy does not [21].

III. ENTROPIC UNCERTAINTY RELATIONS

To obtain entropic relations, we will use a version of Riesz's theorem (see theorem 297 in the book [30]). Let $\mathsf{x} \in \mathbb{C}^n$ be n-tuple of complex numbers x_j and let $\mathsf{T} = [[t_{ij}]]$. Define η to be maximum of $|t_{ij}|$, i.e. $\eta \triangleq \max\{|t_{ij}| : 1 \leq i \leq m, 1 \leq j \leq n\}$. The fixed matrix T describes a linear transformation $\mathbb{C}^n \to \mathbb{C}^m$. That is, to each x we assign m-tuple $\mathsf{y} \in \mathbb{C}^m$ with elements

$$y_i(x) = \sum\nolimits_{j=1}^{n} t_{ij}\, x_j \quad (i = 1, \ldots, m) \ . \tag{3.1}$$

For $b \geq 1$, the l_b norm of vector x is $\|\mathsf{x}\|_b = \Bigl(\sum_j |x_j|^b\Bigr)^{1/b}$. For $\beta \geq 1/2$, we use a like function $\|\mathsf{q}\|_\beta = \Bigl(\sum_j q_j^\beta\Bigr)^{1/\beta}$ of probability distribution $\mathsf{q} = \{q_j\}$, though it is not a norm for $\beta < 1$. Riesz's theorem is formulated as follows.

Lemma III.1. *Suppose the matrix T satisfies $\|\mathsf{y}\|_2 \leq \|\mathsf{x}\|_2$ for all $\mathsf{x} \in \mathbb{C}^n$, $1/a + 1/b = 1$ and $1 < b < 2$; then*

$$\|\mathsf{y}\|_a \leq \eta^{(2-b)/b} \|\mathsf{x}\|_b \ . \tag{3.2}$$

We will now obtain an improved version of the statement emerged in the paper [35]. To each generalized resolution $\mathcal{M} = \{\mathsf{M}_i\}$ and given mixed state ρ, we assign the probabilistic vector $\mathsf{p} = \{p_i\}$ with elements $p_i = \mathrm{tr}(\mathsf{M}_i\rho) \equiv \bigl\|\mathsf{M}_i^{1/2}\sqrt{\rho}\bigr\|_F^2$. Another resolution $\mathcal{N} = \{\mathsf{N}_j\}$ is assigned by the vector q with elements $q_j = \mathrm{tr}(\mathsf{N}_j\rho) \equiv \bigl\|\mathsf{N}_j^{1/2}\sqrt{\rho}\bigr\|_F^2$.

Lemma III.2. *For any two resolutions $\mathcal{M} = \{\mathsf{M}_i\}$ and $\mathcal{N} = \{\mathsf{N}_j\}$ of the identity and given density operator ρ, there holds*

$$\|\mathsf{p}\|_\alpha \leq g(\mathcal{M},\mathcal{N}|\rho)^{2(1-\beta)/\beta}\, \|\mathsf{q}\|_\beta \ , \tag{3.3}$$

where $1/\alpha + 1/\beta = 2$, $1/2 < \beta < 1$ and the function $g(\mathcal{M},\mathcal{N}|\rho)$ is defined by

$$g(\mathcal{M},\mathcal{N}|\rho) \triangleq \max\Bigl\{(p_i q_j)^{-1/2}\, |\mathrm{tr}(\mathsf{M}_i\mathsf{N}_j\rho)| :\ p_i \neq 0,\ q_j \neq 0\Bigr\} \ . \tag{3.4}$$

Proof We first consider the case when both the resolutions $\mathcal{M} = \{\mathsf{M}_i\}$ and $\mathcal{N} = \{\mathsf{N}_j\}$ are orthogonal. For those values of labels i and j that satisfy $\mathrm{tr}(\mathsf{M}_i\rho) \neq 0$ and $\mathrm{tr}(\mathsf{N}_j\rho) \neq 0$, we put (generally non-Hermitian) operators

$$\omega_i = \|\mathsf{M}_i\sqrt{\rho}\|_F^{-1}\, \mathsf{M}_i\sqrt{\rho} \ , \quad \theta_j = \|\mathsf{N}_j\sqrt{\rho}\|_F^{-1}\, \mathsf{N}_j\sqrt{\rho} \ . \tag{3.5}$$

It is clear that $\|\omega_i\|_F = 1$ and $\|\theta_j\|_F = 1$. Since the resolutions $\{\mathsf{M}_i\}$ and $\{\mathsf{N}_j\}$ are orthogonal, we further have $\langle\omega_i\,,\omega_k\rangle_{\mathrm{hs}} = \delta_{ik}$ and $\langle\theta_j\,,\theta_k\rangle_{\mathrm{hs}} = \delta_{jk}$. The matrix elements of transformation T are then defined by

$$t_{ij} = \langle\omega_i\,,\theta_j\rangle_{\mathrm{hs}} = \|\mathsf{M}_i\sqrt{\rho}\|_F^{-1}\|\mathsf{N}_j\sqrt{\rho}\|_F^{-1}\langle\mathsf{M}_i\sqrt{\rho}\,,\mathsf{N}_j\sqrt{\rho}\rangle_{\mathrm{hs}} \ . \tag{3.6}$$

We now rewrite (3.1) as $y_i(x) = \langle\omega_i\,,\sigma\rangle_{\mathrm{hs}}$, where $\sigma = \sum_j x_j\theta_j$ by definition. Due to $\|\sigma\|_F^2 \equiv \sum_j |x_j|^2$ and $\sigma = \sum_i y_i\omega_i + \varpi$, where $\langle\omega_i\,,\varpi\rangle_{\mathrm{hs}} = 0$ for all i, we have $\big\|\sum_i y_i\omega_i\big\|_F^2 \leq \|\sigma\|_F^2$ and the precondition of Lemma III.1 herewith. So we apply (3.2) to the special values $y_i' = \|\mathsf{M}_i\sqrt{\rho}\|_F$ and $x_j' = \|\mathsf{N}_j\sqrt{\rho}\|_F$. By the completeness relation,

$$\mathbb{I}\sqrt{\rho} = \sum\nolimits_k y_k'\omega_k = \sum\nolimits_j x_j'\theta_j \ ,$$

whence $y_i' = \sum_j \langle\omega_i\,,\theta_j\rangle_{\mathrm{hs}}\, x_j'$, i.e. the values y_i' are related to x_j' via the transformation T too. Since the operators M_i and N_j are projective, $p_i = |y_i'|^2$ and $q_j = |x_j'|^2$, whence $\|\mathsf{p}\|_\alpha = \|\mathsf{y}'\|_a^2$ and $\|\mathsf{q}\|_\beta = \|\mathsf{x}'\|_b^2$ with $\alpha = a/2$, $\beta = b/2$. The statement of Lemma III.1 then results in the inequality $\|\mathsf{p}\|_\alpha^{1/2} \leq \max|\langle\omega_i\,,\theta_j\rangle_{\mathrm{hs}}|^{(1-\beta)/\beta}\,\|\mathsf{q}\|_\beta^{1/2}$, under the conditions $1/\alpha + 1/\beta = 2$ and $1/2 < \beta < 1$. Noting that the maximum of modulus of (3.6) is actually $g(\mathcal{M},\mathcal{N}|\rho)$, we resolve the case when the resolutions $\mathcal{M}$ and $\mathcal{N}$ are both orthogonal. To generalize (3.3) to the case of arbitrary two resolutions, we will use the method proposed in [36] and further developed in [35]. In the extended space $\mathcal{H}\oplus\mathcal{K}$, the resolutions $\mathcal{M}$ and $\mathcal{N}$ are realized as new resolutions $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ respectively, and the $\widetilde{\mathcal{M}}$ is now orthogonal by Naimark's extension (see, e.g., Sect. 5.1 in [26]). It can be made in such a way that for any density matrix ρ on $\mathcal{H}$, $\mathrm{tr}(\mathsf{M}_i\rho) = \mathrm{tr}(\widetilde{\mathsf{M}}_i\widetilde{\rho})$, $\mathrm{tr}(\mathsf{N}_j\rho) = \mathrm{tr}(\widetilde{\mathsf{N}}_j\widetilde{\rho})$, and $\mathrm{tr}(\mathsf{M}_i\mathsf{N}_j\rho) = \mathrm{tr}(\widetilde{\mathsf{M}}_i\widetilde{\mathsf{N}}_j\widetilde{\rho})$ (the $\widetilde{\rho}$ is built from ρ by adding zero rows and columns). Hence we have the same values of entropies [35] and $g(\mathcal{M},\mathcal{N}|\rho) = g(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}|\widetilde{\rho})$. By a similar extension of $\widetilde{\mathcal{N}}$, the question is quite reduced to the above case of two orthogonal resolutions. $\square$

Using simple algebra (see the proof of proposition 3 in [35]), the inequality (3.3) can be rewritten as

$$R_\alpha(\mathcal{M}|\rho) + R_\beta(\mathcal{N}|\rho) \geq -2\ln g(\mathcal{M},\mathcal{N}|\rho) \ . \tag{3.7}$$

For a pure state, this relation in terms the Rényi entropies coincides with the one deduced in the previous work [35]. With respect to the spectral decomposition $\rho = \sum_\lambda \lambda\,\boldsymbol{\psi}_\lambda\boldsymbol{\psi}_\lambda^\dagger$, we define

$$f(\mathcal{M},\mathcal{N}|\rho) = \max\Big\{\big(p_i^{(\lambda)}q_j^{(\lambda)}\big)^{-1/2}|\langle\mathsf{M}_i\boldsymbol{\psi}_\lambda\,,\mathsf{N}_j\boldsymbol{\psi}_\lambda\rangle| :\ p_i^{(\lambda)} \neq 0,\ q_j^{(\lambda)} \neq 0\Big\} \ , \tag{3.8}$$

where the probabilities $p_i^{(\lambda)} = \big\|\mathsf{M}_i^{1/2}\boldsymbol{\psi}_\lambda\big\|_2^2$ and $q_j^{(\lambda)} = \big\|\mathsf{N}_j^{1/2}\boldsymbol{\psi}_\lambda\big\|_2^2$. The above inequality with $f(\mathcal{M},\mathcal{N}|\rho)$ instead of $g(\mathcal{M},\mathcal{N}|\rho)$ was obtained in [35]. For an impure state, we have $g(\mathcal{M},\mathcal{N}|\rho) < f(\mathcal{M},\mathcal{N}|\rho)$ and a stronger bound in (3.7). For canonically conjugate variables, the uncertainty relations in terms of Rényi's entropies were given in [37].

Let us proceed to a relation with the Tsallis entropies. The sum $H_\alpha(p_i) + H_\beta(q_j)$ cannot be arbitrarily small because of the constraint (3.3). We rewrite this sum as

$$H_\alpha(p_i) + H_\beta(q_j) = \phi(\xi,\zeta) = \frac{\xi - 1}{1-\alpha} + \frac{\zeta - 1}{1-\beta} \tag{3.9}$$

in terms of the variables $\xi = \sum_i p_i^\alpha = \|\mathsf{p}\|_\alpha^\alpha$, $\zeta = \sum_j q_j^\beta = \|\mathsf{q}\|_\beta^\beta$ and the function $\phi(\xi,\zeta)$. Assuming $\alpha > 1 > \beta$, we obviously have $\xi \leq 1$ and $\zeta \geq 1$. Adding (3.3) in the form $\zeta \geq \gamma\,\xi^{\beta/\alpha}$, where $\gamma = g(\mathcal{M},\mathcal{N}|\rho)^{-2(1-\beta)}$, we have arrived at a task of minimizing $\phi(\xi,\zeta)$ under the above conditions. It is important here that $g(\mathcal{M},\mathcal{N}|\rho) \leq 1$ and, therefore, $\gamma \geq 1$. Indeed, the Cauchy-Schwarz inequality for the Hilbert-Schmidt inner product directly shows that the modulus of (3.6) does not exceed one. The task is solved in Appendix A, and the sum $H_\alpha(p_i) + H_\beta(q_j)$ cannot be less than

$$\phi(\xi_0,1) = \frac{\gamma^{-\alpha/\beta} - 1}{1-\alpha} = \frac{g(\mathcal{M},\mathcal{N}|\rho)^{2\alpha(1-\beta)/\beta} - 1}{1-\alpha} = \frac{g(\mathcal{M},\mathcal{N}|\rho)^{2(\alpha-1)} - 1}{1-\alpha} = \ln_\alpha\Big(g(\mathcal{M},\mathcal{N}|\rho)^{-2}\Big) \ ,$$

where we used $(1-\beta)/\beta = (\alpha-1)/\alpha$ due to $1/\alpha + 1/\beta = 2$. By $g(\mathcal{M},\mathcal{N}|\rho) = g(\mathcal{N},\mathcal{M}|\rho)$, we claim the following.

Theorem III.3. *For any two resolutions $\mathcal{M} = \{\mathsf{M}_i\}$ and $\mathcal{N} = \{\mathsf{N}_j\}$ of the identity and given density operator ρ, there holds*

$$H_\alpha(\mathcal{M}|\rho) + H_\beta(\mathcal{N}|\rho) \geq \ln_\mu\Big(g(\mathcal{M},\mathcal{N}|\rho)^{-2}\Big) \ , \tag{3.10}$$

where $1/\alpha + 1/\beta = 2$ and $\mu = \max\{\alpha,\beta\}$.

The inequality (3.10) gives the uncertainty relation in terms of the Tsallis entropies for two generalized measurements. In spirit and origin, it is like to the relation via the Rényi entropies (3.7). Note that relations in terms of the Tsallis entropies have been considered for particular cases of the position and momentum [38, 39] and the spin-1/2

components [40, 41]. It is of some interest to get a state-independent version of (3.10). Since the function $\ln_\mu x$ is increasing for $\mu > 0$, $g(\mathcal{M},\mathcal{N}|\rho) \leq f(\mathcal{M},\mathcal{N}|\rho)$ and [35]

$$f(\mathcal{M},\mathcal{N}|\boldsymbol{\psi}) \leq \max\Bigl\{ \bigl\| \mathsf{M}_i^{1/2}\mathsf{N}_j^{1/2} \bigr\|_\infty :\ \mathsf{M}_i \in \mathcal{M},\ \mathsf{N}_j \in \mathcal{N} \Bigr\} \triangleq \bar{f}(\mathcal{M},\mathcal{N}) \ , \tag{3.11}$$

we have the state-independent bound

$$H_\alpha(\mathcal{M}|\rho) + H_\beta(\mathcal{N}|\rho) \geq \ln_\mu\Bigl(\bar{f}(\mathcal{M},\mathcal{N})^{-2} \Bigr) \ . \tag{3.12}$$

Note that $\bar{f}(\mathcal{M},\mathcal{N}) \leq 1$ is provided by $\bigl\| \mathsf{M}_i^{1/2}\mathsf{N}_j^{1/2} \bigr\|_\infty^2 \leq \|\mathsf{M}_i\|_\infty \|\mathsf{N}_j\|_\infty \leq 1$. Here the inequality on the left is a Cauchy-Schwarz inequality for ordinary matrix products and the spectral norm (see, e.g., the result (4.50) in [42]) and the inequality on the right follows from the completeness relation. The inequality $\bar{f}(\mathcal{M},\mathcal{N}) \leq 1$ is saturated if and only if for some two elements M_0, N_0 and nonzero $\boldsymbol{\varphi}_0 \in \mathcal{H}$ there hold $\mathsf{M}_0\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0$ and $\mathsf{N}_0\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0$ simultaneously. In other words, $\bar{f}(\mathcal{M},\mathcal{N}) = 1$ implies that there exist those operators $\mathsf{M}_0 \in \mathcal{M}$ and $\mathsf{N}_0 \in \mathcal{N}$ that act as commuting projectors in a nonempty subspace $\mathcal{K}_0 \subset \mathcal{H}$. Otherwise, the entropic bound in the relation (3.12) is nontrivial. So, we have extended the relation, conjectured in [43] and later proved in [8], to the Tsallis entropies and general quantum measurements. Let us consider two concrete examples of specific interest.

Example III.4. The first example is a pair of complementary observables in a d-level system (for the Rényi formulation, see [37]). Let complex amplitudes $\tilde{c}_k$ and c_l be connected by the discrete Fourier transform

$$\tilde{c}_k = \frac{1}{\sqrt{d}} \sum\nolimits_{l=1}^{d} e^{2\pi \mathrm{i} k\, l/d}\, c_l \ , \tag{3.13}$$

and the corresponding probability distributions $p_k = |\tilde{c}_k|^2$ and $q_l = |c_l|^2$. The transformation (3.13) is the "canonical" example that leads to complementary observables [43]. It follows from $\|\tilde{\mathsf{c}}\|_2 = \|\mathsf{c}\|_2$ and (3.2) that

$$\|\mathsf{c}\|_a \leq \left(\frac{1}{\sqrt{d}} \right)^{(2-b)/b} \|\tilde{\mathsf{c}}\|_b \ , \qquad \|\tilde{\mathsf{c}}\|_a \leq \left(\frac{1}{\sqrt{d}} \right)^{(2-b)/b} \|\mathsf{c}\|_b \ , \tag{3.14}$$

where $1/a + 1/b = 1$ and $1 < b < 2$. Squaring, we get $\|\mathsf{q}\|_\alpha \leq (1/d)^{(1-\beta)/\beta}\|\mathsf{p}\|_\beta$, $\|\mathsf{p}\|_\alpha \leq (1/d)^{(1-\beta)/\beta}\|\mathsf{q}\|_\beta$ under the conditions on α and β from Lemma III.2. In its symmetric form, the uncertainty relation in terms of the Tsallis entropies is written as $H_\alpha(p_k) + H_\beta(q_l) \geq \ln_\mu d$, where $1/\alpha + 1/\beta = 2$ and $\mu = \max\{\alpha,\beta\}$.

Example III.5. The angle ϕ and the angular momentum J_z can similarly be treated. Taking one and the same size $\delta\varphi$ for all the angular bins (i.e. the ratio $2\pi/\delta\varphi$ is a strictly positive integer), we introduce probabilities

$$p_k = \int_{k\delta\varphi}^{(k+1)\delta\varphi} d\varphi\, |\Psi(\varphi)|^2 \ , \qquad q_l = |c_l|^2 \ , \tag{3.15}$$

where the coefficients c_l's are related to the expansion $\Psi(\varphi) = (2\pi)^{-1/2} \sum_{l=-\infty}^{+\infty} c_l \exp(\mathrm{i}\, l\, \varphi)$, with respect to the eigenstates of the J_z. Using theorem 192 of the book [30] for integral means and assuming $\beta < 1 < \alpha$, we have

$$\frac{1}{\delta\varphi} \int_{k\delta\varphi}^{(k+1)\delta\varphi} d\varphi\, |\Psi(\varphi)|^{2\beta} \leq \left(\frac{1}{\delta\varphi} \int_{k\delta\varphi}^{(k+1)\delta\varphi} d\varphi\, |\Psi(\varphi)|^2 \right)^\beta \tag{3.16}$$

and the inversed inequality with α instead of β. Summing these inequalities with respect to k and then raising them to the powers $1/\beta$ and $1/\alpha$ respectively, we finally write

$$\|\Psi\|_b^2 \leq \delta\varphi^{(1-\beta)/\beta} \|\mathsf{p}\|_\beta \ , \qquad \delta\varphi^{(1-\alpha)/\alpha} \|\mathsf{p}\|_\alpha \leq \|\Psi\|_a^2 \ , \tag{3.17}$$

where the norm $\|\Psi\|_b = \left(\int_0^{2\pi} d\varphi\, |\Psi(\varphi)|^b \right)^{1/b}$ and $b = 2\beta$, $a = 2\alpha$. Combining the relations (3.17) with the Young-Hausdorff inequalities (see, e.g., section 8.17 in [30]), which are viewed in our notation as

$$\|\mathsf{c}\|_a \leq \left(\frac{1}{\sqrt{2\pi}} \right)^{(2-b)/b} \|\Psi\|_b \ , \qquad \|\Psi\|_a \leq \left(\frac{1}{\sqrt{2\pi}} \right)^{(2-b)/b} \|\mathsf{c}\|_b \ , \tag{3.18}$$

we get $\|\mathsf{q}\|_\alpha \leq \bigl(\delta\varphi/2\pi\bigr)^{(1-\beta)/\beta} \|\mathsf{p}\|_\beta$, $\|\mathsf{p}\|_\alpha \leq \bigl(\delta\varphi/2\pi\bigr)^{(1-\beta)/\beta} \|\mathsf{q}\|_\beta$. So, there holds $H_\alpha(\varphi) + H_\beta(\mathsf{J}_z) \geq \ln_\mu\bigl(2\pi/\delta\varphi\bigr)$, where $1/\alpha + 1/\beta = 2$ and $\mu = \max\{\alpha,\beta\}$. When $\mu \to 1$, this new inequality in terms of Tsallis' entropies coincides with the relation in terms of Rényi's entropies deduced in [37].

Finally, we consider the case of extremal unravelings within the general formulation (3.10). For the extremal unravelings $\mathcal{A}_\rho^{(ex)}$ and $\mathcal{B}_\rho^{(ex)}$ of the super-operators $\$_A$ and $\$_B$, we obtain the entropic uncertainty relation

$$H_\alpha(\mathcal{A}_\rho^{(ex)}|\rho) + H_\beta(\mathcal{B}_\rho^{(ex)}|\rho) \geq \ln_\mu\Big(g\big(\mathcal{A}_\rho^{(ex)}, \mathcal{B}_\rho^{(ex)}|\rho\big)^{-2}\Big) \ , \tag{3.19}$$

where the $g(\mathcal{A}_\rho^{(ex)}, \mathcal{B}_\rho^{(ex)}|\rho)$ is put by (3.4) with $\mathsf{M}_i = \mathsf{A}_i^\dagger \mathsf{A}_i$, $\mathsf{N}_j = \mathsf{B}_j^\dagger \mathsf{B}_j$ in terms of Kraus operators $\mathsf{A}_i \in \mathcal{A}_\rho^{(ex)}$, $\mathsf{B}_j \in \mathcal{B}_\rho^{(ex)}$. We can also write the uncertainty relation for unravelings extremal with respect to the Rényi entropies. Using (3.7), for $\alpha > 1$ we obtain the relation

$$R_\alpha\big(\mathcal{A}'^{(ex)}_{\alpha,\rho}|\rho\big) + R_\beta\big(\mathcal{B}_\rho^{(ex)}|\rho\big) \geq -2\ln g\big(\mathcal{A}'^{(ex)}_{\alpha,\rho}, \mathcal{B}_\rho^{(ex)}|\rho\big) \ , \tag{3.20}$$

where $\mathcal{A}'^{(ex)}_{\alpha,\rho}$ denotes the unraveling of $\$_A$ extremal for the Rényi entropy of order $\alpha > 1$. This unraveling differs from the one given by (2.9) and also depends on the parameter α in general. A search of explicit analytic expression for $\mathcal{A}'^{(ex)}_{\alpha,\rho}$ seems to be complicated, because concavity (convexity) things cannot be used here. Nevertheless, the entropic relation in terms of the Rényi entropies holds, as a mathematical inequality at least.

IV. CONCLUSION

We have considered the unraveling (i.e. the concrete set of Kraus operators) of a super-operator that is extremal with respect to all the Tsallis entropies of a positive order and the Rényi entropies of an order $0 < \alpha < 1$. This general result is formally posed in Theorem II.1. If one of unravelings is given explicitly, then this minimizing unraveling is easily calculated by diagonalizing a certain Hermitian matrix. The known relation between the Shannon entropy of an ensemble of pure states and the von Neumann entropy of the risen density operator is extended to both the Tsallis and Rényi entropies of the mentioned orders. The case of Tsallis entropy allows further extension to a mixture of density operators (see the bounds (2.19)). Due to Riesz's theorem, there exists an inequality between certain functions of the probability distributions generated by two resolutions of the identity (see Lemma III.2). This inequality gives an origin for the uncertainty relations, given by (3.7) for the Rényi entropies and by Theorem III.3 for the Tsallis entropies. The general formulation (3.10) is a new result in the topic. It has been illustrated within the two interesting cases (see Examples III.4 and III.5). In the formulae (3.19) and (3.20), both the entropic uncertainty relations are naturally recast for the extremal unravelings of two given trace-preserving super-operators.

Appendix A: A minimum of the function

To obtain a lower bound on the sum of Tsalis entropies, we find the minimal value of the function (3.9) in the domain D such that $0 \leq \xi \leq 1$, $1 \leq \zeta < \infty$, $\zeta \geq \gamma\,\xi^{\beta/\alpha}$. When $\gamma > 1$, the curve $\zeta = \gamma\,\xi^{\beta/\alpha}$ cuts off the down right corner of the rectangle $\{(\xi,\zeta) :\ 0 \leq \xi \leq 1,\ 1 \leq \zeta < \infty\}$ and herewith the point $(1,1)$ in which $\phi = 0$ (see Fig. 1). In the interior of D, we have

$$\frac{\partial\phi}{\partial\xi} = \frac{1}{1-\alpha} < 0 \ , \qquad \frac{\partial\phi}{\partial\zeta} = \frac{1}{1-\beta} > 0 \ , \tag{A1}$$

due to $\alpha > 1 > \beta$. So the minimum is reached on the boundary of the domain D. Using (A1), the task is merely reduced to minimizing $\phi(\xi,\zeta)$ on segment of the curve $C :\ \zeta = \gamma\,\xi^{\beta/\alpha}$ between the point $(\xi_0, 1)$, where $\xi_0 = \gamma^{-\alpha/\beta}$, and the point $(1,\gamma)$. Substituting $\zeta = (\xi/\xi_0)^{\beta/\alpha}$ in (3.9) and differentiating with respect to ξ, we obtain

$$\frac{1}{1-\alpha} + \frac{\beta}{\alpha(1-\beta)\xi}\left(\frac{\xi}{\xi_0}\right)^{\beta/\alpha} = \frac{1}{\alpha-1}\left(\frac{1}{\xi}\left(\frac{\xi}{\xi_0}\right)^{\beta/\alpha} - 1\right) \ , \tag{A2}$$

where we used $\beta/(1-\beta) = \alpha/(\alpha-1)$ because of $1/\alpha + 1/\beta = 2$. When $\xi_0 < 1$, the quantity (A2) is strictly positive for $\xi_0 \leq \xi \leq 1$ and the minimal value is $\phi(\xi_0, 1) = (\xi_0 - 1)/(1-\alpha)$ too.

[1] Heisenberg W 1927 *Z. Phys.* **43** 172

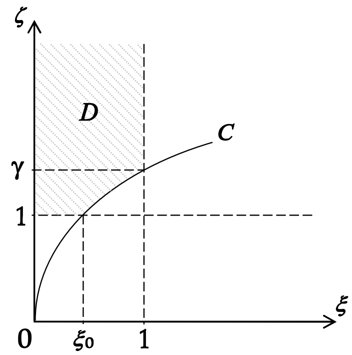

FIG. 1: The domain D in which we find the minimum of the function $\phi(\xi,\zeta)$ defined by (3.9).

[2] Hall M J W 1999 *Phys. Rev.* A **59** 2602
[3] Busch P, Heinonen T and Lahti P J 2007 *Phys. Rep.* **452** 155
[4] Robertson H P 1929 *Phys. Rev.* **34** 163
[5] Lynch R 1995 *Phys. Rep.* **256** 357
[6] Barnett S M and Pegg D T 1989 *J. Mod. Optics* **36** 7
[7] Hirschman I I 1957 *Am. J. Math.* **79** 152
[8] Maassen H and Uffink J B M 1988 *Phys. Rev. Lett.* **60** 1103
[9] Wehner S and Winter A 2010 *New J. Phys.* **12** 025009
[10] Mamojka B 1974 *Int. J. Theor. Phys.* **11** 73
[11] Deutsch D 1983 *Phys. Rev. Lett.* **50** 631
[12] Massar S and Spindel P 2008 *Phys. Rev. Lett.* **100** 190401
[13] Urizar-Lanz I and Tóth G 2010 *Phys. Rev. A* **81** 052108
[14] Ivanovic I D 1992 *J. Phys. A: Math. Gen.* **25** L363
[15] Sanchez J 1993 *Phys. Lett. A* **173** 233
[16] Hall M J W 2008 *J. Phys. A: Math. Theor.* **41** 255301
[17] Man'ko M A 2009 *J. Russ. Laser Res.* **30** 514
[18] Kraus K 1983 *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (*Lecture Notes in Physics* vol 190) (Berlin: Springer)
[19] Il'ichev L V 2003 *JETP* **96** 982
[20] http://tsallis.cat.cbpf.br/biblio.htm
[21] Aczél J and Daróczy Z 1975 *On Measures of Information and their Characterizations* (New York: Academic Press)
[22] Tsallis C 1988 *J. Stat. Phys.* **32** 479
[23] Havrda J and Charvát F 1967 *Kybernetika* **3** 30
[24] Rényi A 1961 *On measures of entropy and information*, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, pp. 547-561 (University of California Press: Berkeley–Los Angeles)
[25] Życzkowski K 2003 *Open Sys. Inf. Dyn.* **10** 297
[26] Watrous J 2008 *CS 798: Theory of Quantum Information* (University of Waterloo: http://www.cs.uwaterloo.ca/∼watrous/quant-info/lecture-notes/all-lectures.pdf)
[27] Holevo A S 1982 *Probabilistic and Statistical Aspects of Quantum Theory* (*North-Holland Series in Statistics and Probability* vol 1) (Amsterdam: North-Holland)
[28] Preskill J 1998 *Lecture Notes for Physics 229: Quantum Computation and Information* (California Institute of Technology: http://www.theory.caltech.edu/people/preskill/ph229/)
[29] Carmichael H J 1993 *An Open Systems Approach to Quantum Optics* (*Lecture Notes in Physics* vol m18) (Berlin: Springer)
[30] Hardy G H, Littlewood J E and Polya G 1934 *Inequalities* (London: Cambridge University Press)
[31] Breslin J K and Milburn G J 1997 *J. Mod. Optics* **44** 2469
[32] Hughston L P, Jozsa R and Wootters W K 1993 *Phys. Lett. A* **183** 14
[33] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
[34] Rastegin A E 2010 Fano type quantum inequalities in terms of q-entropies *E-print* arXiv:1010.1811 [quant-ph]
[35] Rastegin A E 2010 *J. Phys. A: Math. Theor.* **43** 155302
[36] Krishna M and Parthasarathy K R 2002 *Sankhya, Ser. A* **64** 842
[37] Bialynicki-Birula I 2006 *Phys. Rev.* A **74** 052101
[38] Rajagopal A K 1995 *Phys. Lett.* A **205** 32
[39] Wilk G and Włodarczyk Z 2009 *Phys. Rev.* A **79** 062108
[40] Majernik V and Majernikova E 2001 *Rep. Math. Phys.* **47** 381
[41] Majernik V, Majernikova E and Shpyrko S 2003 *Cent. Eur. J. Phys.* **3** 393
[42] Zhan X 2002 *Matrix Inequalities* (Berlin: Springer)
[43] Kraus K 1987 *Phys. Rev.* D **35** 3070